LINES SIMULTANEOUSLY BISECTING THE PERIMETER AND AREA OF A TRIANGLE

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ABSTRACT. We present a ruler and compass construction and a simple enumeration of lines simultaneously bisecting the perimeter and area of a given triangle. If the lengths of the sides are \(a \leq b \leq c\), and \(s\) is the semiperimeter, the number of such lines is 0, 1, or 2 according as \((s - b)^2 + (s - c)^2 - (s - a)^2\) is negative, zero, or positive. We also construct triangles with integer sides for which these perimeter-area bisectors intersect the sides at point at integer (or half-integer) distances from the vertices.

1. INTRODUCTION

The problem of enumeration of lines simultaneously bisecting the perimeter and area of a given triangle has been studied in several recent articles. In this note we give a ruler and compass construction and a simple enumeration of such lines (Theorem 3 below). These are called equalizers in [2] etc and B-lines in [4]. We shall use the term perimeter-area bisectors.

Given triangle \(ABC\) with sidelengths \(BC = a\), \(CA = b\), and \(AB = c\), let \(s\) denote the semiperimeter. Let \(Y\) and \(Z\) be points on the sides \(AC\) and \(AB\) respectively. We call the line \(YZ\)

(i) a perimeter bisector in angle \(A\) if \(AY + AZ = ZB + BC + CY = s\),

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\[\text{Key words and phrases.} \quad \text{Perimeter-area bisectors, integer triangles.}\]
(ii) an area bisector in angle $A$ if the areas of triangle $AZY$ and convex quadrilateral $ZBCY$ are each one half of the area of triangle $ABC$,

(iii) a perimeter-area bisector in angle $A$ if it is both a perimeter bisector and an area bisector in angle $A$.

A perimeter-area bisector $YZ$ in angle $A$ of triangle $ABC$ depends on the lengths of $AY = y$ and $AZ = z$. We require $y + z = s$ and $yz = \frac{1}{2}bc$. These two lengths are the roots of the quadratic polynomial

$$Q_a(t) := t^2 - st + \frac{1}{2}bc = \left( t - \frac{s}{2} \right)^2 - \frac{D_a}{4},$$

where

$$D_a = s^2 - 2bc = ((s - a) + (s - b) + (s - c))^2 - 2((s - c) + (s - a))((s - a) + (s - b)) = (s - b)^2 + (s - c)^2 - (s - a)^2.$$

It is interesting to note that a perimeter bisector necessarily passes through the incenter of the triangle ([2,4]). We shall make use of this to simplify a construction in §5 below. Clearly, if the line $YZ$ is a perimeter-area bisector in angle $A$, then so is its reflection in the bisector of angle $A$, provided that the reflections $Z'$ of $Y$, and $Y'$ of $Z$ are on the sides $AB$ and $AC$ respectively. The line $Y'Z'$ in Figure 6, for example, is not a perimeter-area bisector in angle $A$.

2. Construction of Perimeter-Area Bisectors in Angle $A$

Figure 2 shows a simple geometric construction of these two lengths.

Suppose there is a circle $C$ which has

(i) a chord through $A$ divided into two segments of lengths $b$ and $\frac{1}{2}c$, and

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{}
\end{figure}
(ii) another chord of length s.
Then a tangent through A to the concentric circle $C'$ tangent to the chord in (ii) solves our problem, because it clearly has the same length s, and by the intersecting chords theorem, the product of the two segments divided by $A$ is the same as the product of the segments of the chord in (i).

Now, given triangle $ABC$, we take $C$ to be the circumcircle of $A'B'C'$, where $A'$ is a point on the extension of $CB$ such that $CA' = s$, and $B'$ is a point on the extension of $CA$ such that $AB' = \frac{s}{2}$. (The point $A'$ is the point of tangency of the line $BC$ with the excircle on the opposite side of $C$.) The circle $C'$ is concentric with $C$, and tangent to $A'C$ at its midpoint.

If the vertex $A$ is on or outside the circle $C'$, construct a tangent from $A$ to $C'$ to intersect $C$ the circle $C$ at $B''$ and $C''$. With these, construct points $Y$ on $AC$, $Z'$ on $AB$ such that $AY = AZ' = AB''$, and $Y'$ on $AC$, $Z$ on $AB$ such that $AY' = AY = AC''$. Each of the lines $YZ$ and $Y'Z'$ is a perimeter-area bisector in angle $A$, provided both of its endpoints are on the sides $AC$ and $AB$ respectively. For example, in Figure 2, the line $YZ$ is a perimeter-area bisector, but $Y'Z'$ is not.

Clearly, if the vertex $A$ is inside the circle $C'$, there is no perimeter-area bisector in angle $A$.

We also speak of perimeter-area bisectors in angles $B$ and $C$. In \S 3, 4 below, we determine the number of perimeter-area bisector in terms of the lengths $a$, $b$, $c$.

3. ISOSCELES TRIANGLES
We begin with the equilateral triangle. Clearly each angle bisector is a perimeter-area bisector. There are no others since for the perimeter-area bisectors in angle $A$, the polynomial $Q_a(t) = (t - a)(t - \frac{s}{2})$ leads to the bisectors of angle $B$ and $C$.

**Lemma 1.** A perimeter-area bisector passes through a vertex of the triangle if and only if it is isosceles.

**Proof.** If $AB = AC$, then the bisector of angle $A$ is clearly a perimeter-area bisector in angle $B$ and $C$. Conversely, let a perimeter-area bisector $YZ$ in angle $A$ pass through a vertex, say $Y = C$. Then, $b$ is a root of the polynomial $Q_a(t)$. Since $Q_a(b) = \frac{1}{2}b(b - a)$, we must have $b = a$, and the triangle is isosceles. □

Now, let $ABC$ be isosceles with $AB = AC$. We determine the perimeter-area bisectors in angle $A$. Since $b = c$, in this case,

$$Q_a(t) = (t - a + 2b) - \frac{D_a}{4},$$

where $D_a = \frac{(a+2b)^2 - 8b^2}{4}$.

If $D_a < 0$, there is no perimeter-area bisector in angle $A$.

If $D_a = 0$, then $Q_a(t)$ has a unique positive root $\frac{a+2b}{4} < b$. In this case, there is a unique perimeter-area bisector in angle $A$.

Suppose $D_a > 0$ so that $Q_a(t)$ has two positive roots.
(i) If $a > b$, then $Q_a(b) = \frac{1}{2}b(b - a) < 0$, and $Q_a(t)$ has a root greater than $b$. In this case, there is no perimeter-area bisector in angle $A$.
(ii) If $a < b$, then since $0 < \frac{s}{2} < b$, and

$$Q_a(0) > 0, \quad Q_a(\frac{s}{2}) = -\frac{D_a}{4} < 0, \quad Q_a(b) > 0,$$
both roots of $Q_a$ are smaller than $b$. In this case, there are two perimeter-area bisectors in angle $A$.

Counting also the bisector of angle $A$, we obtain the following enumeration of perimeter-area bisectors of an isosceles triangle.

**Proposition 2.** Let $ABC$ be an isosceles triangle with $b = c$, the number of perimeter-area bisectors is

1. if $D_a < 0$ or $a > b$,
2. if $D_a = 0$,
3. if $D_a > 0$ and $a \leq b$.

**Remark.** In terms of $a$ and $b = c$, $D_a$ is negative, zero, or positive according as $a$ is greater than, equal to, or less than $2(\sqrt{2} - 1)b$.

### 4. SCALENE TRIANGLES

Since we have dealt with the isosceles case, we shall assume $a < b < c$.

If $D_a < 0$, then $Q_a(t)$ has no real root, and there is no perimeter-area bisector in angle $A$.

If $D_a = 0$, then $Q_a(t)$ has a double root $\frac{s}{2}$, which is smaller than $b$. (Proof: $\frac{s}{2} = \frac{a+b+c}{4} < \frac{a+b}{2} < b$). Therefore, there is a unique perimeter-area bisector in angle $A$. This is the perpendicular to the bisector of angle $A$ at the incenter.

If $D_a > 0$, the two positive roots of $Q_a(t)$ are smaller than $b$ (and $c$) since

- $Q_a(0) = \frac{1}{2}bc > 0$,
- $Q_a\left(\frac{s}{2}\right) = -\frac{D_a}{4} < 0$,
- $Q_a(b) = \frac{1}{2}b(b-a) > 0$.

In this case, there are two perimeter-area bisectors.

Now consider the perimeter-area bisector in angle $B$. These are given by the roots of the quadratic polynomial

$$Q_b(t) = t^2 - st + \frac{1}{2}ca = \left(t - \frac{s}{2}\right)^2 - \frac{D_b}{4},$$

where

$$D_b = (s - c)^2 + (s - a)^2 - (s - b)^2 > 0.$$

In this case,

- $Q_b(0) = \frac{1}{2}ca > 0$,
- $Q_b(a) = -\frac{1}{2}a(b-a) < 0$,
- $Q_b(c) = \frac{1}{2}c(c-b) > 0$.

The two roots are smaller than $c$, but are separated by $a$. There is only one perimeter-area bisector in the “middle” angle $B$.

Finally, the perimeter-area bisectors in angle $C$ are given by the roots of the quadratic polynomial

$$Q_c(t) = t^2 - st + \frac{1}{2}ab = \left(t - \frac{s}{2}\right)^2 - \frac{D_c}{4},$$

where

$$D_c = (s - a)^2 + (s - b)^2 - (s - c)^2 > 0.$$

In this case,

- $Q_c(0) = \frac{1}{2}ab > 0$,
- $Q_c(a) = -\frac{1}{2}a(c-a) < 0$,
- $Q_c(b) = -\frac{1}{2}b(c-b) < 0$.

The larger root is greater than $b$ (and $a$). There is no perimeter-area bisector in the largest angle.
Combining these results with Proposition 2, we obtain the following enumeration result on the number of perimeter-area bisectors.

**Theorem 3.** Let \( ABC \) be a triangle with \( a \leq b \leq c \). The number of perimeter-area bisectors is 1, 2, or 3 according as
\[
D_a = (s - b)^2 + (s - c)^2 - (s - a)^2
\]
is negative, zero, or positive.

5. **Triangles with exactly two perimeter-area bisectors**

Here is a construction of triangles satisfying \( D_a = 0 \). Let \( X \) be a point on a given a segment \( BC \). Construct
(i) a point \( PY \) on the perpendicular to \( BC \) at \( B \) such that \( BP = XC \),
(ii) the circle, center \( X \), radius \( XP \), to intersect the line \( BC \) at \( B' \) and \( C' \) (so that \( B \) and \( B' \) are on the same side of \( X \), as are \( C \) and \( C' \)),
(iii) the circles, center \( B \), radius \( BC' \), and center \( C \), radius \( CB' \), to intersect at \( A \).

\( ABC \) is a triangle whose incircle touches \( BC \) at \( X \), and satisfies
\[
(s - b)^2 + (s - c)^2 = (s - a)^2.
\]
The perimeter-area bisector in angle \( A \) is the perpendicular to the bisector at the incenter \( I \) (see [2, Lemma 1] and Figure 3).

**Figure 3.**

**Remark.** If in (ii) above, the radius of the circle is taken to be shorter (respectively longer) than \( XP \), then with \( A \) constructed in (iii), the number of perimeter-area bisectors in angle \( A \) is 2 (respectively 0).

There is a unique right triangle satisfying \( D_a = 0 \). If we put \( t = \tan \frac{A}{2} \), then
\[
a : b : c = 2t : 1 - t^2 : 1 + t^2, \quad \text{and} \quad D_a = (t + 1)(2t^3 - 2t^2 + 3t - 1).\]
This has a unique real root, which is approximately 0.396608 \cdots. Correspondingly, \( A \approx 43.2674 \) degrees.

6. **Integer Triangles**

We determine scalene triangles with integer sides whose perimeter-area bisectors are given by points on the sides with rational (integer or half-integer) distances from the vertices.
6.1. **Triangles with** $D_a < 0$. We require $D_b = (s-c)^2 + (s-a)^2 - (s-b)^2 = v^2$ for an integer $v$.

Let $d_1 = h_1^2 + k_1^2$ with $h_1 > k_1$ and $d_2 = h_2^2 + k_2^2$ with $h_2 > k_2$. We further assume $k_1 h_2 < h_1 k_2 < 3k_1 h_2$. The product $d_1 d_2$ is a sum of two squares in two ways, as $d_1 d_2 = p^2 + q^2 = u^2 + v^2$ with

\[ p = h_1 h_2 + k_1 k_2, \quad q = h_1 k_2 - k_1 h_2; \]
\[ u = h_1 h_2 - k_1 k_2, \quad v = h_1 k_2 + k_1 h_2. \]

Note that $p > u > q$. Also,

\[ u^2 + q^2 - p^2 = h_1^2 k_2^2 + h_2^2 k_1^2 - 6h_1 h_2 k_1 k_2 \]
\[ < 2h_1^2 k_2^2 - 6h_1 h_2 k_1 k_2 \]
\[ = 2h_1 k_2 (h_1 k_2 - 3h_2 k_1) \]
\[ < 0. \]

Therefore, by setting $s - a = p$, $s - b = u$, $s - c = q$, we have $D_a < 0$ and $D_b = (s-c)^2 + (s-a)^2 - (s-b)^2 = v^2$.

For example, by using the five smallest Pythagorean triples, we obtain the following integer triangles $(a,b,c)$ with only one perimeter-area bisectors $X_b Z_b$ in angle $B$ with $(BX_b, BZ_b)$ given in the rightmost column in the table below.

<table>
<thead>
<tr>
<th>$(d_1,d_2)$</th>
<th>$(h_1,k_1)$</th>
<th>$(h_2,k_2)$</th>
<th>$(p,q)$</th>
<th>$(u,v)$</th>
<th>$(a,b,c,s)$</th>
<th>$(BX_b,BZ_b)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5,29)</td>
<td>(4,3)</td>
<td>(21,20)</td>
<td>(144,17)</td>
<td>(24,143)</td>
<td>(41,161,168,185)</td>
<td>(21,164)</td>
</tr>
<tr>
<td>(13,5)</td>
<td>(12,5)</td>
<td>(4,3)</td>
<td>(63,16)</td>
<td>(33,56)</td>
<td>(49,79,96,112)</td>
<td>(28,84)</td>
</tr>
<tr>
<td>(13,17)</td>
<td>(12,5)</td>
<td>(15,8)</td>
<td>(220,21)</td>
<td>(140,171)</td>
<td>(161,241,360,381)</td>
<td>(105,276)</td>
</tr>
<tr>
<td>(13,29)</td>
<td>(12,5)</td>
<td>(21,20)</td>
<td>(352,135)</td>
<td>(152,345)</td>
<td>(287,487,504,639)</td>
<td>(147,492)</td>
</tr>
<tr>
<td>(17,5)</td>
<td>(15,8)</td>
<td>(4,3)</td>
<td>(84,13)</td>
<td>(36,77)</td>
<td>(49,97,120,133)</td>
<td>(28,105)</td>
</tr>
<tr>
<td>(17,29)</td>
<td>(15,8)</td>
<td>(21,20)</td>
<td>(475,132)</td>
<td>(155,468)</td>
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<td>(147,615)</td>
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<tr>
<td>(25,5)</td>
<td>(24,7)</td>
<td>(4,3)</td>
<td>(117,44)</td>
<td>(75,100)</td>
<td>(119,161,192,236)</td>
<td>(68,168)</td>
</tr>
<tr>
<td>(25,13)</td>
<td>(24,7)</td>
<td>(12,5)</td>
<td>(323,36)</td>
<td>(253,204)</td>
<td>(289,359,576,612)</td>
<td>(204,408)</td>
</tr>
<tr>
<td>(25,17)</td>
<td>(24,7)</td>
<td>(15,8)</td>
<td>(416,87)</td>
<td>(304,297)</td>
<td>(391,503,720,807)</td>
<td>(255,552)</td>
</tr>
</tbody>
</table>

6.2. **Triangles with** $D_a = 0$. In this case, $(s - b)^2 + (s - c)^2 = (s - a)^2$. It is enough to determine $(a,b,c)$ from a Pythagorean triple. In this case, $AY_a = AZ_a = s$. Note that $D_b = 2(s - c)^2$ and the perimeter-area bisector in angle $B$ cannot intersect the sides $BC$ and $BA$ with rational length $BX_b$ and $BZ_b$. Here are some small examples.

<table>
<thead>
<tr>
<th>$(s-a,s-b,s-c)$</th>
<th>$(a,b,c)$</th>
<th>$s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5,4,3)</td>
<td>(7,8,9)</td>
<td>12</td>
</tr>
<tr>
<td>(13,12,5)</td>
<td>(17,18,25)</td>
<td>30</td>
</tr>
<tr>
<td>(17,15,8)</td>
<td>(23,25,32)</td>
<td>40</td>
</tr>
<tr>
<td>(25,24,7)</td>
<td>(31,32,49)</td>
<td>56</td>
</tr>
<tr>
<td>(29,21,20)</td>
<td>(41,49,50)</td>
<td>70</td>
</tr>
</tbody>
</table>

6.3. **Triangles with** $D_a > 0$. We require $D_a$ and $D_b$ to be squares of integers $u$ and $v$. Let $D := (s - a)^2 + (s - b)^2 + (s - c)^2$. Then $D = u^2 + 2(s - a)^2 = v^2 + 2(s - b)^2$. A number $D$ can be written as $u^2 + 2a^2$ in two different ways if
and only if it is divisible by two prime numbers congruent to 1 or 3 (mod 8). Let \( D \) be one such number with
\[
D = u_1^2 + 2x^2 = v^2 + 2y^2, \quad x > y.
\]
If, in addition, \( D - x^2 - y^2 = z^2 \) for an integer \( z < y \), then by setting \( s - a = x, \ s - b = y, \ s - c = z \), we obtain \( (a, b, c) = (y + z, z + x, x + y) \) for which \( D_a = u^2 \) and \( D_b = v^2 \). The table below shows all possibilities with squarefree \( D < 10000 \), where we reduce \( (a, b, c) \) be a factor \( \frac{1}{2} \) when \( x, y, z \) are all odd.

<table>
<thead>
<tr>
<th>( D )</th>
<th>( (x, y, z) )</th>
<th>( (a, b, c) )</th>
<th>( \frac{s-a}{2} )</th>
<th>( \frac{s+b}{2} )</th>
<th>( \frac{s-c}{2} )</th>
<th>( \frac{s+d}{2} )</th>
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<tr>
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<td>(21, 22, 30)</td>
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<tr>
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<td>(63, 48)</td>
<td>(28, 105)</td>
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<td></td>
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</table>

![Figure 4](image1.png)  
![Figure 5](image2.png)
REFERENCES


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