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EQUALIZERS IN RIGHT TRIANGLES

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ABSTRACT. A triangle equalizer is a line bisecting both its area and perimeter. This work provides a detailed account of equalizer locations in a right triangle, showing that there exist triangles with exactly one, two or three equalizers, but no more. Right triangles with exactly two equalizers are quite rare: there are specific values for right triangles with exactly two equalizers. The methodology based on the equations is known as Pythagorean Triples. Although the fact that constitute a geometrical issue, calculations based on methods of Analysis.

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1. INTRODUCTION

An interesting question from classical Euclidean geometry concerns the existence of lines called ”equalizers” that bisect both the area and the perimeter of a region. In this work it is considered the possibility of a straight line bisecting a right triangle, its position on the triangle and the number of such lines. Also, such a line is called *triangle equalizer*. The main contribution of this paper is that it is proved that there are right triangles with either *one*, or *two* equalizers, although the latter seem to be a “rare” category of right triangles. This work is an original, complete and thorough study of the equalizers for any right triangle. The study of this geometric topic is made in an analytical way, as the precise mathematical expressions to solve the issue are given. The existence of the number of equalizers is determined by the different values taken by the sides of the right triangle. At this point, is stressed that this paper refers to the branch of mathematics known as the ‘Geometry of the Triangle’, which flourished mainly in the 19^{th} century.
The reason for such a mathematical search was an article by George Berzsenyi [2], in which he conjectured that in each triangle there are either only one or only three equalizers. At the same time, there is made no effort to solve the problem, but the author encourages readers to explore it. Besides, H. Bailey states that there are no triangles with exactly two equalizers. Furthermore, the article of Kdokostas Dimitrios [3] provides a detailed account of the location of the equalizers in a triangle. An interesting lemma in that article states that any equalizer of a triangle goes through the incenter of the triangle, a line through that incenter is an area divider if and only if it is a perimeter splitter and the equalizers of a triangle are the area dividers through its center.

2. ABOUT PYTHAGOREAN TRIPLES

Consider a line ℓ at the surface of a right triangle K_1YK_2. We will call ℓ *equalizer* of the triangle K_1YK_2, when it is dividing the triangle into two equiareal and equiperimetric parts. In this paper, we will study the case of the right triangle with hypotenuse $v = K_1K_2$ and two vertical sides $\kappa_1 = YK_2$ and $\kappa_2 = YK_1$ (figure 1). There will be explored questions such as that of the existence of equalizers for a right triangle and their number. For a rectangular triangle is valid the Pythagorean Theorem, namely: $v^2 = \kappa_1^2 + \kappa_2^2$. Also, each triangle with sides equal to: $(dv,\, d\kappa_1,\, d\kappa_2)$, where $d \in \mathbb{R}_+^*$, is a right triangle, in so far the Pythagorean Theorem is satisfied [1]. It is also known that the *Pythagorean Triples* of a right triangle (i.e., the right triangle where the sides are integers) are given by the following general transformation:

$$(v,\, \kappa_1,\, \kappa_2) = (m^2 + n^2,\, 2mn,\, m^2 - n^2),$$

where: $m > n$ and $m,\, n \in \mathbb{N}^*$. To define a Pythagorean Triple, it is sufficient to establish that one having the greatest common divisor of three positive integers equal to the unit (*Primitive Pythagorean Triples*). This general transformation can be written as follows:

$$(v,\, \kappa_1,\, \kappa_2) = \big[n^2(\frac{m^2}{n^2} + 1),\, n^2\frac{2m}{n},\, n^2(\frac{m^2}{n^2} - 1)\big], \tag{1}$$

or, otherwise:

$$(v,\, \kappa_1,\, \kappa_2) = \big[m^2(1 + \frac{n^2}{m^2}),\, m^2\frac{2n}{m},\, m^2(1 - \frac{n^2}{m^2})\big], \tag{2}$$

For our analysis, we consider that the relations (1) and (2) apply for m and $n \in \mathbb{N}^*$, with $m > n > 0$ (either $0 > n > m$), i.e. it is considered that the relations (1) and (2) are valid for two consecutive sizes m and n (and not two discrete ones, such as it is the set of natural numbers). If for the case (1) we set: $r = \dfrac{m}{n}$, with $r > 1$, and taking into account the observation that each triad values: $(dv,\, d\kappa_1,\, d\kappa_2)$ is consisting a right triangle, we have the transformation:

$$(v,\, \kappa_1,\, \kappa_2) = (r^2 + 1,\, 2r,\, r^2 - 1).$$

Also, if for the case (2) we set: $r = \dfrac{n}{m}$, with $0 < r < 1$, we have the transformation:

$$(v,\, \kappa_1,\, \kappa_2) = (1 + r^2,\, 2r,\, 1 - r^2).$$

In summary, in our analysis will consider the transformation:

$$(v,\, \kappa_1,\, \kappa_2) = \begin{cases} (r^2 + 1,\, 2r,\, r^2 - 1), & r > 1 \\ (1 + r^2,\, 2r,\, 1 - r^2), & 1 > r > 0 \end{cases} \tag{3}$$

The two transformations in (3) are equivalent and each of them can be chosen for the search of equalizers in a right triangle. The transformation (3) will help for a more complete study of the problem, as it is leading the studying factors as functions of an independent variable r and well known methods of Analysis can simplify the problem considerably.

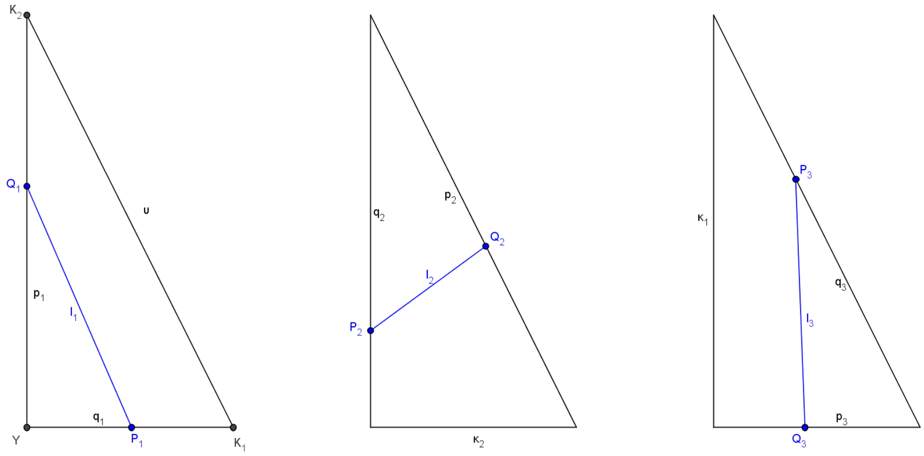

FIGURE 1. Right Triangles

3. NOTES ABOUT THE EQUALIZER'S PROBLEM

In summary, for the analysis of the problem there are three possible arrangements of the equalizer ℓ in the right triangle K_1YK_2, which are presented briefly in the following paragraphs. If: $0 \leq p_1 \leq \kappa_2$, $0 \leq q_1 \leq \kappa_1$, $0 \leq p_2 \leq v$, $0 \leq q_2 \leq \kappa_2$, $0 \leq p_3 \leq \kappa_1$ and $0 \leq q_3 \leq v$, then we have schematically in **Figure 1**: Also, it is considered the coefficients: $k_1 = s^2 - 2\kappa_1\kappa_2$, $k_2 = s^2 - 2v\kappa_2$ and $k_3 = s^2 - 2v\kappa_1$, where s is the semiperimeter of the right triangle. The coefficients k_1, k_2, and k_3 generated in the analysis of the problem when, in the mathematical application of the definition of equalizer for the right triangle in the above figure, the following relationships are derived:

$$(2p_1 - s)^2 = (s^2 - 2\kappa_1\kappa_2), \quad (2p_2 - s)^2 = (s^2 - 2v\kappa_2)$$

$$(2p_3 - s)^2 = (s^2 - 2v\kappa_1).$$

If $k_1 \geq 0$, the the two possible pairs (p_1, q_1), which formed the existence of the equalizer, are:

$$(p_1, q_1) = [\frac{1}{2}(s \pm \sqrt{k_1}), \frac{1}{2}(s \mp \sqrt{k_1})]. \tag{4}$$

The equalizer, if existing has the form: $\ell_1 = 2\sqrt{\frac{s(s-v)}{\kappa_1\kappa_2}(p_1^2 - p_1 s + \frac{\kappa_1\kappa_2 s}{4(s-v)})}$. Also, if $k_2 \geq 0$ and $k_3 \geq 0$ then the possible pairs (p_2, q_2) and (p_3, q_3), which formed the existence of the equalizer, are:

$$(p_2, q_2) = [\frac{1}{2}(s \pm \sqrt{k_2}), \frac{1}{2}(s \mp \sqrt{k_2})] \tag{5}$$

$$(p_3, q_3) = [\frac{1}{2}(s \pm \sqrt{k_3}), \frac{1}{2}(s \mp \sqrt{k_3})] \tag{6}$$

The equalizers, if existing, have the following forms:

$$\ell_2 = 2\sqrt{\frac{s(s-\kappa_1)}{v\kappa_2}(p_2^2 - p_2 s + \frac{v\kappa_2 s}{4(s-\kappa_1)})}, \quad \ell_3 = 2\sqrt{\frac{s(s-\kappa_2)}{v\kappa_1}(p_3^2 - p_3 s + \frac{v\kappa_1 s}{4(s-\kappa_2)})}.$$

4. FIRST TRANSFORMATION ($1 < r$)

The coefficients k_1, k_2 and k_3, as well as the semiperemeter s of the right triangle and the equalizers ℓ_2 and ℓ_3, receive the following forms:

$$k_1 = (r^4 - 3r^3 + r^2 + 4r), \quad k_2 = (-r^4 + 2r^3 + r^2 + 2)$$

$$k_3 = (r^4 - 2r^3 + r^2 - 4r), \quad \ell_2 = \sqrt{r^2(r+1)(3-r)},$$

$$\ell_3 = \sqrt{r(r+1)^2(r-2)}, \quad s = r(r+1).$$

The coefficient k_1 is positive if: $r \in (1, \infty)$. The coefficient k_2 is positive if: $r \in (1, r_2)$, where:

$$r_2 = \frac{\sqrt[3]{3}}{3}(\sqrt[3]{9} + \sqrt[3]{9+\sqrt{78}} + \sqrt[3]{9-\sqrt{78}}),$$

or, otherwise, with an accuracy of six significant digits: $r_2 \cong 2.52138$. Also, the coefficient k_3 is positive if: $r \in (r_3, \infty)$, where:

$$r_3 = \frac{1}{3}(2 + \sqrt[3]{53+6\sqrt{78}} + \sqrt[3]{53-6\sqrt{78}},$$

or, otherwise, with an accuracy of six significant digits: $r_3 \cong 2.31459$

5. SECOND TRANSFORMATION ($1 > r > 0$)

The coefficients k_1, k_2 and k_3, as well as the semiperemeter s of the right triangle and the equalizers ℓ_2 and ℓ_3, receive the following forms:

$$k_1 = (4r^3 + r^2 - 2r + 1), \quad k_2 = (2r^4 + r^2 + 2r - 1)$$

$$k_3 = (-4r^3 + r^2 - 2r + 1), \quad \ell_2 = \sqrt{(r+1)(3r-1)},$$

$$\ell_3 = \sqrt{(r+1)^2(1-2r)}, \quad s = (1+r).$$

The coefficient k_1 is positive if: $r \in (0,\ 1)$ and the coefficient k_2 is positive if: $r \in (r_2,\ 1)$, where:

$$r_2 = \frac{1}{6}(2 + \sqrt[3]{8+6\sqrt{78}} - \sqrt[3]{-8+\sqrt{78}}),$$

or, otherwise, with an accuracy of six significant digits: $r_2 \cong 0.396608$. Also, the coefficient k_3 is positive if: $r \in (0,\ r_3)$, where:

$$r_3 = \frac{1}{12}(1 + \sqrt[3]{181+24\sqrt{78}} - \sqrt[3]{-181+24\sqrt{78}}),$$

or, otherwise, with an accuracy of six significant digits: $r_3 \cong 0.432041$. The main conclusions presented in the following paragraphs.

6. CONCLUDING REMARKS

Based on those outlined in the above paragraphs, essentially the study of the number of equalizers for a right triangle has degenerated to investigate the values by the parameter r. Each element of the triangle (sides, semiperimeter, coefficients and equalizer) was expressed as a function of the independent variable r and greatly facilitate the resolution of the matter. The conclusions are summarized as follows in Figure 2:

Values of $r \in (1, +\infty)$	**Number of equalizer**	**Type of equalizer**
$(1, r'_3)$	1	ℓ'_2
r'_3	2	$\ell'_2 \cong 3.49$ and $\ell'_3 \cong 2.83$
(r'_3, ρ_1)	3	ℓ'_2, ℓ'_3 and ℓ'_3
$[\rho_1, r'_2)$	3	ℓ'_2, ℓ'_2 and ℓ'_3
r'_2	2	$\ell'_2 \cong 3.27$ and $\ell'_3 \cong 4.04$
$(r'_2, +\infty)$	1	ℓ'_3

Values of $r \in (0, 1)$	**Number of equalizer**	**Type of equalizer**
$(0, r''_2)$	1	ℓ''_3
r''_2	2	$\ell''_2 \cong 0.515$ and $\ell''_3 \cong 0.635$
$(r''_2, \rho_2]$	3	ℓ''_2, ℓ''_2 and ℓ''_3
(ρ_2, r''_3)	3	ℓ''_2, ℓ''_3 and ℓ''_3
r''_3	2	$\ell''_2 \cong 0.651$ and $\ell''_3 \cong 0.528$
$(r''_3, 1)$	1	ℓ''_2

Figure 2

where: $\rho_1 = (\sqrt{2} - 1)$ and $\rho_2 = (\sqrt{2} + 1)$. For each of the above ranges of the parameter r, mathematical expressions of all elements of the triangle (sides, semiperimeter, coefficients and equalizers) are exposed in sections 4 and 5. When encoding the conclusions, the following theorem is presented.

Theorem 1. *Each right triangle has either one, or two, or three equalizers.*

At this point, the issues concerning equiareal and equiperimetric parts of a right triangle can be extended and generalized for different convex flat shapes. The concept of the equalizer, for example, may be configured based on the following definition:

Definition 1. ***Equalizer*** *is any line that divides a convex flat geometric shape into two equiareal and equiperimetric sections.*

There is always an Equalizer for any body and that is a fact came up from a useful Topology theorem: the *Ham-Sandwich Theorem*, also called the '*Stone=Tukey Theorem*' (after Arthur H. Stone and John W. Tukey). The theorem states that, given $d \geq 2$ measurable solids in $\mathbb{R}^d$, it is possible to bisect all of them in half with a single $(d-1)-$dimensional hyperplane. In other words, the Ham-Sandwich Theorem states something like the following ordinary language proposition: take a sandwich made of a slice of ham and two slices of bread. No matter where one places the pieces of the sandwich in the kitchen, or house, or universe, so long as one's knife is long enough one can cut all three pieces in half in only one pass. Proving the theorem for $d = 2$ (where it is known as the '*Pancake Theorem*') is simple and can be found in Courant and Robbins (1978) [4].

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